

XXZ spin chain in a transverse field as a regularization of the sine-Gordon model

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We consider here the XXZ spin chain perturbed by the operator σ^x ("in a transverse field") which is a lattice regularization of the sine-Gordon model. This can be shown using conformal perturbation theory. We calculate the mass ratios of particles which lie in a discrete part of the spectrum and obtain results in accord with the DHN formula and in disagreement with recent calculations in the literature based on the numerical Bethe ansatz and infinite momentum frame methods. We also analyze the short distance behavior of these states (UV or conformal limit). Our result for conformal dimension of the second breather state is different from that conjectured by Klassen and Melzer [Int. J. Mod. Phys. A **8**, 4131 (1993)] and is consistent with this paper for other states. [S0556-2821(99)04110-7]

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I. INTRODUCTION

The sine-Gordon (SG) and massive Thirring (MT) models in two dimensions belong to a group of the most studied quantum field theories (QFT's) and are certainly the best understood nontrivial massive field theories. A large number of different techniques have been successfully tested on these models and they led us to a number of interesting results, including the famous duality relation between them [1–3].

Regarding a mass spectrum, we can classify all methods into basically three groups: (a) the semiclassical Dashen-Hasslacher-Neveu (DHN) method [4], (b) factorized scattering theory [5], and (c) methods based on the Bethe ansatz, which can be further subdivided into continuum [6,7] and discrete ones [8,9] (some lattice regularizations were used). The results of all these methods were the same; besides the soliton and antisoliton (fermion and antifermion in MTM language) there are bound states (breathers) and their masses are given by

$$m_n = 2m \sin \frac{n\pi\beta^2}{2(8\pi - \beta^2)}, \quad n = 1, 2, \dots < \frac{8\pi}{\beta^2} - 1, \quad (1.1)$$

where m is the soliton mass and β is the coupling constant in the SG model (SGM) [see Eq. (2.1)]. Because of Coleman's theorem of the equivalence between the SGM and the MTM in the soliton number (charge) zero sector (proved using perturbative expansion in mass), the same spectrum should be valid for the MTM. Using standard conventions (as in [1]), a connection ("duality relation") between β and the MTM coupling constant g_0 (in the Schwinger normalization) is given by

$$1 + \frac{g_0}{\pi} = \frac{4\pi}{\beta^2}.$$

However, recently it has been claimed [10–12] that the mass spectrum of the MTM is different than Eq. (1.1) and that there is only one breather in the whole interval $g_0 > 0$ [for negative values of g_0 fermion and antifermion repel

each other and there are no bound states, like in Eq. (1.1)]. In [10], using the infinite momentum frame technique and working only in $q\bar{q}$ sector of the Fock space (neglecting $qq\bar{q}\bar{q}$ and higher fermion components), authors obtained the mass of the (only) breather:

$$M = 2m \cos \alpha, \quad (1.2)$$

where the parameter $0 \leq \alpha < \pi/2$ is obtained by solving the following equation:

$$\frac{\tan \alpha}{\frac{\pi}{2} - \alpha} = \frac{g}{\pi} \left[1 + \frac{1}{\cos^2 \alpha} \left(1 - \frac{g}{4\pi} \right) \right]$$

and g is the MTM coupling constant in Johnson's normalization which is connected to that in Schwinger's normalization by

$$g_0 = \frac{2g}{2 - \frac{g}{\pi}}.$$

Afterwards, in [11] the authors reexamined an analysis of [6], but contrary to [6] they numerically solved Bethe ansatz equations for a finite space extension and a finite number of quasiparticles, and after that made an extrapolation to infinity. Their analysis confirmed results of [10]; they found only one breather, with the mass in good agreement with Eq. (1.2).

In this paper we propose ourselves to calculate certain properties of the SGM like mass ratios and scaling dimensions of operators creating particle states. Using the conformal perturbation theory [13,14] it can be shown that the XXZ spin chain with an even number of sites and periodic boundary conditions in a transverse magnetic field (σ^x perturbation) is spin chain regularization of the SGM (see Appendix B in [14]). We numerically diagonalize the spin chain Hamiltonian up to 16 sites and extrapolate results to the infinite length continuum limit using the BST extrapolation algorithm [15,16]. The same method was previously applied

to conformal unitary models perturbed by some relevant (usually thermal) operator [17–19]. In this way we can obtain estimates of mass ratios without further assumptions, particularly those criticized in [10–12].

Results of our analysis are as follows. For a whole range of the coupling constants we can cover ($0 < \beta \leq \sqrt{2\pi}$) our results agree with the DHN formula (1.1) and disagree with Eq. (1.2), i.e., results of [10,11]. Of course, we could not say anything about breathers higher than third because they lie in a continuum part of the spectrum ($m_n > 2m_1$ for $n \geq 3$). We should also say that precision in this method is far from that achieved by, e.g., Bethe ansatz methods, so we cannot claim that the DHN formula is exact.

Finally, as a byproduct, we studied the UV limit of particle states. It agrees with that conjectured in [14] for (anti)soliton and first breather. However, for the second breather we obtain the same scaling dimension as for the first, contrary to [14].

II. THE SGM AS A MASSIVE PERTURBATION OF THE GAUSSIAN MODEL

The SGM is a $(1+1)$ -dimensional field theory of a pseudoscalar field φ , defined classically by the Lagrangian:

$$\mathcal{L}_{SG} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \lambda \cos(\beta \varphi). \quad (2.1)$$

Here λ is a mass scale (with mass dimension depending on a regularization scheme), β is a dimensionless coupling (which does not renormalize) and one identifies field configurations that differ by a period $2\pi/\beta$ of the potential (because we want to have “ordinary” QFT with a unique vacuum).

In [14] it was shown that SGM can be viewed as a perturbed conformal field theory (CFT) when the second term in Eq. (2.1) is treated as a (massive) perturbation. We will now repeat here relevant results of their analyses.

An unperturbed theory $\lambda = 0$ (approached in UV limit) is the free massless compactified pseudoscalar CFT (known as Gaussian model). It is conventional to use $\Phi \equiv \sqrt{\pi} \varphi$, so that the radius of compactification r , defined by equivalence $\Phi \sim \Phi + 2\pi r$ is connected to β with

$$r = \frac{\sqrt{\pi}}{\beta}. \quad (2.2)$$

Solution of the equation of motion in Euclidean space, $\partial \bar{\partial} \Phi(z, \bar{z}) = 0$, is

$$\Phi(z, \bar{z}) = \frac{1}{2} (\phi + \bar{\phi}).$$

The Gaussian model is a CFT with central charge $c = 1$ and an operator algebra generated by the primary fields $V_{m,n}$

$$V_{m,n} = :e^{i(m/r)\Phi(z, \bar{z}) + i2nr\bar{\Phi}(z, \bar{z})}:, \quad (2.3)$$

where $\bar{\Phi} \equiv (\phi - \bar{\phi})/2$. Conformal dimensions of $V_{m,n}$ are

$$(\Delta_{m,n}, \bar{\Delta}_{m,n}) = \left(\frac{1}{2} \left(\frac{m}{2r} + nr \right)^2, \frac{1}{2} \left(\frac{m}{2r} - nr \right)^2 \right) \quad (2.4)$$

so that its scaling dimension and (Lorentz) spin are

$$d_{m,n} = \Delta_{m,n} + \bar{\Delta}_{m,n} = \left(\frac{m}{2r} \right)^2 + (nr)^2 = \frac{m^2 \beta^2}{4\pi} + \frac{n^2 \pi}{\beta^2},$$

$$s_{m,n} = \Delta_{m,n} - \bar{\Delta}_{m,n} = mn.$$

It is understood that $V_{m,n}$ are normalized so that

$$\langle V_{m,n}(z, \bar{z}) V_{m',n'}(0,0) \rangle = \delta_{m,-m'} \delta_{n,-n'} z^{-2\Delta_{m,n}} \bar{z}^{-2\bar{\Delta}_{m,n}}.$$

Because of $V_{m,n}^\dagger = V_{-m,-n}$, we can define Hermitian combinations

$$V_{m,n}^{(+)} \equiv \frac{1}{2} (V_{m,n} + V_{-m,-n}),$$

$$V_{m,n}^{(-)} \equiv \frac{i}{2} (V_{-m,-n} - V_{m,n})$$

which will be useful later.

In [14] it is argued that an UV limit of the SGM is generated by

$$L_b = \{V_{m,n} | m, n \in \mathbf{Z}\}. \quad (2.5)$$

We suppose that Hilbert space of the full (perturbed) theory is isomorphic to that of the unperturbed theory. From Eqs. (2.2) and (2.3) follows that a (properly normalized) perturbing operator in the SGM (2.1) is

$$\cos(\beta \varphi) = V_{1,0}^{(+)} \quad (2.6)$$

which means that λ has mass dimension $y = 2 - d_{1,0} = 2 - \beta^2/4\pi$. From the condition of relevancy of the perturbation, i.e., $y > 0$, we obtain Coleman’s bound $\beta^2 < 8\pi$. Also, from Eqs. (2.5) and (2.6) we can see that SGM has $\tilde{U}(1) \times Z_2 \times \tilde{Z}_2$ internal symmetry group. The $\tilde{U}(1)$ acts as a shift on Φ , i.e., $V_{m,n} \rightarrow e^{ian} V_{m,n}$, while Z_2 and \tilde{Z}_2 are generated by $R: (\Phi, \bar{\Phi}) \rightarrow (-\Phi, \bar{\Phi})$ (i.e., $V_{m,n} \rightarrow V_{-m,n}$) and $\tilde{R}: (\Phi, \bar{\Phi}) \rightarrow (\Phi, -\bar{\Phi})$ (i.e., $V_{m,n} \rightarrow V_{m,-n}$), respectively.

To conclude this section, consider the SGM defined on a cylinder with infinite time dimension and space extension equal to L . There are three independent constants with which we can express all quantities in the theory, β , λ and L with mass dimensions $d_\beta = 0$, $d_\lambda = 2 - d_{1,0} = 2 - \beta^2/(4\pi)$ and $d_L = -1$. It is useful to define the dimensionless scaling parameter μ ,

$$\mu \equiv \lambda L^{d_\lambda} = \lambda L^{2 - \beta^2/4\pi}, \quad (2.7)$$

and use β , μ and λ as a set of independent parameters. Now, from ordinary dimensional analysis follows that any quantity X in the theory, with mass dimension d_X , can be written as

$$X = \lambda^{d_X/d_\lambda} g_X(\beta, \mu) = \lambda^{d_X/(2 - \beta^2/4\pi)} g_X(\beta, \mu), \quad (2.8)$$

where g_X is the scaling function connected to X . We see that all dimensionless quantities depend only on β and μ . Especially, we have, for masses of particles,

$$m_i(\beta, \mu, \lambda) = \lambda^{(2-\beta^2/4\pi)^{-1}} G_i(\beta, \mu). \quad (2.9)$$

Now, there are two interesting limits. The first one is the infinite length limit, $L \rightarrow \infty$, which is equal to $\mu \rightarrow \infty$ [see Eq. (2.7)]. We are interested here in mass ratios:

$$r_i(\beta) = \lim_{\mu \rightarrow \infty} \frac{m_{i+1}(\beta, \mu, \lambda)}{m_1(\beta, \mu, \lambda)} = \lim_{\mu \rightarrow \infty} \frac{G_{i+1}(\beta, \mu)}{G_1(\beta, \mu)}.$$

The second interesting limit is the UV limit given by $L \rightarrow 0$ ($\mu \rightarrow 0$). Basic assumption of conformal perturbation theory is that the perturbed QFT should approach CFT smoothly in the UV limit. It means that if we write Eq. (2.8) in the form

$$X = X_{CFT}(\beta, L) + \lambda^{d_X/d_\lambda} h_X(\beta, \mu), \quad (2.10)$$

where X_{CFT} is the value for X in the conformal point ($\lambda = 0$), then a Taylor expansion for $\mu^{d_X/d_\lambda} h_X(\beta, \mu)$ around $\mu = 0$ will have finite radius of convergence and $h_X(\beta, 0) = 0$. Specifically, for the mass gaps we have the well-known formula

$$(m_i)_{CFT} = \frac{2\pi}{L} d_i,$$

where d_i is the scaling dimension of the operator which creates that state from the vacuum. Now from Eqs. (2.9), (2.10) and (2.7) follows

$$\begin{aligned} m_i(\beta, \mu, \lambda) &= \frac{2\pi}{L} d_i + \lambda^{1/d_\lambda} H_i(\beta, \mu) \\ &= \lambda^{1/d_\lambda} [2\pi d_i \mu^{-1/d_\lambda} + H_i(\beta, \mu)] \\ &= \lambda^{(2-\beta^2/4\pi)^{-1}} [2\pi d_i \mu^{-(2-\beta^2/4\pi)^{-1}} + H_i(\beta, \mu)]. \end{aligned} \quad (2.11)$$

Now, what are scaling dimensions of zero-momentum one-particle states in SGM, i.e., of soliton, antisoliton and breathers? In Table I we show values conjectured in [14]. In Sec. V we will show that we obtain a different result for the second breather.

III. SPIN CHAIN REGULARIZATION OF THE SGM

It was proposed (Appendix B in [14]) that the XXZ spin chain with periodic boundary conditions in a transverse magnetic field defined by the Hamiltonian

$$\begin{aligned} H &= - \sum_{n=1}^N (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z + h \sigma_n^x), \\ \vec{\sigma}_{N+1} &\equiv \vec{\sigma}_1, \end{aligned} \quad (3.1)$$

TABLE I. Scaling dimensions of particle states in SGM as conjectured in [14].

State	Operator	Scaling dimension
soliton	$V_{0,1}$	$\frac{\pi}{\beta^2}$
antisoliton	$V_{0,-1}$	$\frac{\pi}{\beta^2}$
p th breather	$V_{p,0}^{[(-)^p]}$	$\frac{p^2 \beta^2}{4\pi}$

where σ_a are Pauli matrices, N is an even integer, $-1 \leq \Delta < 1$ (we use the usual parametrization $\Delta = -\cos \gamma, 0 \leq \gamma < \pi$), is a spin chain regularization of the SGM. The argument has two steps; first, one must show that unperturbed theories are equivalent, i.e., that Eq. (3.1) with $h=0$ is a spin chain regularization of L_b CFT (2.5), and, second, that in the unperturbed theory ($h=0$) perturbation operator σ_n^x is a lattice regularization of $V_{1,0}^{(+)}(x)$.

For a first step one must take $h=0$ in Eq. (3.1), i.e., to consider periodic XXZ spin chain

$$\begin{aligned} H_{XXZ} &= - \sum_{n=1}^N (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z), \\ \vec{\sigma}_{N+1} &\equiv \vec{\sigma}_1, \end{aligned} \quad (3.2)$$

H_{XXZ} commutes with $S^z = 1/2 \sum_{n=1}^N \sigma_n^z$. We denote eigenvalues of S^z by Q . Q is integer (half-odd integer) when N is even (odd) and $-N/2 \leq Q \leq N/2$. H_{XXZ} is also translation-invariant where translations by one site are generated by

$$T = \prod_{n=1}^{N-1} \frac{1}{2} (\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} + 1) \quad (3.3)$$

and we define the (lattice) momentum operator by $T = \exp(-iP)$. From Eq. (3.3) follows that $T^N = 1$, so eigenvalues P_k of the lattice momentum P are given by

$$P_k = \frac{2\pi}{N} k, \quad k=0, 1, \dots, N-1. \quad (3.4)$$

Obviously, P_k are defined mod 2π .

Now, in [20,21] it has been shown that energy-momentum spectrum of the periodic XXZ chain in charge sector Q has the following asymptotic form for large N :

$$E_{Q,\nu}^{n,\bar{n}} = N e_\infty + \frac{2\pi\zeta}{N} \left(\Delta_{Q,\nu}^n + \bar{\Delta}_{Q,\nu}^{\bar{n}} - \frac{c}{12} \right), \quad (3.5a)$$

$$P_{Q,\nu}^l = \frac{2\pi}{N} (\Delta_{Q,\nu}^n - \bar{\Delta}_{Q,\nu}^{\bar{n}}) + \pi \kappa_{Q,\nu}, \quad (3.5b)$$

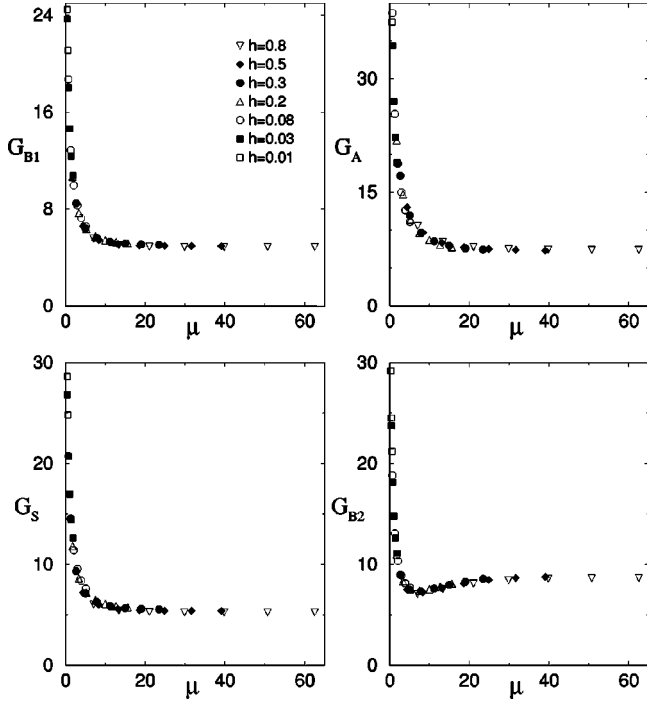


FIG. 1. Scaling functions $\tilde{G}_a(\beta, \mu)$ for the isolated gaps of Hamiltonian (3.1) at $\Delta = -0.9$ (or $\beta^2 = 5.38$). A legend in upper left figure applies to all figures in this article.

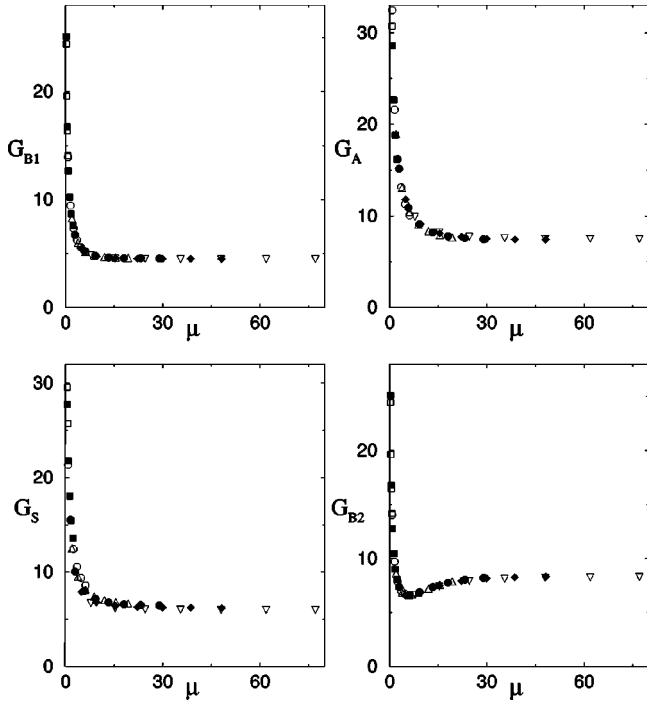


FIG. 2. The same as Fig. 1 but now for $\Delta = -0.6$ (or $\beta^2 = 4.43$).

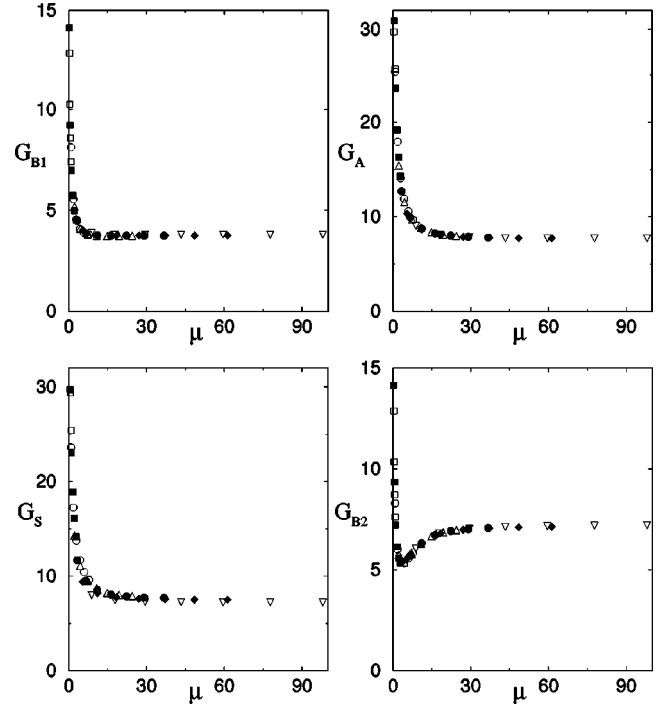


FIG. 3. The same as Fig. 1 but now for $\Delta = -0.1$ (or $\beta^2 = 3.34$).

where $\nu \in \mathbb{Z}$, $n, \bar{n} \in \mathbb{N}^0$, central charge $c = 1$, $\kappa_{Q,\nu} \in \{0, 1\}$, and conformal dimensions $\Delta_{Q,\nu}^n$ and $\bar{\Delta}_{Q,\nu}^{\bar{n}}$ are given by

$$(\Delta_{Q,\nu}^n, \bar{\Delta}_{Q,\nu}^{\bar{n}}) = \left(\frac{1}{2} \left[\frac{Q}{2r} + r\nu \right]^2 + n, \frac{1}{2} \left[\frac{Q}{2r} - r\nu \right]^2 + \bar{n} \right), \quad (3.6)$$

where the compactification radius is $r = [2(1 - \gamma/\pi)]^{-1/2}$. From Eqs. (3.5a), (3.5b), and (3.6) we can infer that the continuum limit of H_{XXZ} defined by

$$H_{XXZ}^{cont} \equiv \frac{1}{\zeta} \lim_{\substack{N \rightarrow \infty \\ a \rightarrow 0}} \frac{1}{a} (H_{XXZ} - N e_\infty), \quad (3.7a)$$

$$P^{cont} \equiv \lim_{\substack{N \rightarrow \infty \\ a \rightarrow 0}} \frac{1}{a} (P - \pi \kappa) \quad (3.7b)$$

(a is lattice constant and $L = Na$ is kept fixed) defines $c = 1$ CFT, and in fact contains L_b of the Gaussian model as we shall see. In Eq. (3.7b) κ is an operator which project states having “nonuniversal macroscopic momentum,” equal to π (see [22]). We shall comment more on this at the end of this section. ζ is a normalization factor and e_∞ is (c-number) nonuniversal bulk energy density. Nonuniversal quantities are subtracted in the QFT limit.

Let us see how one can obtain L_b and L_f from H_{XXZ}^{cont} . First, from Eq. (3.6) it is obvious that

$$(\Delta_{Q,\nu}^0, \bar{\Delta}_{Q,\nu}^0) = (\Delta_{Q,\nu}, \bar{\Delta}_{Q,\nu}),$$

TABLE II. Estimates for the scaled gaps $\tilde{G}_a(\beta, \infty)$ as a function of h at $\Delta = -0.9$ ($\beta^2 = 5.38$). The numbers in brackets give the estimated uncertainty in the last given digit.

h	\tilde{G}_{B1}	\tilde{G}_S	\tilde{G}_A	\tilde{G}_{B2}
0.8	4.85922(5)	5.2274(1)	7.358(2)	8.706(6)
0.5	4.9421(7)	5.368(1)	7.25(1)	8.93(3)
0.3	5.012(6)	5.49(1)	7.10(5)	9.2(1)
0.2	5.04(2)	5.55(3)	6.9(1)	8.7(2)

where $\Delta_{m,n}$ and $\bar{\Delta}_{m,n}$ are conformal dimensions (2.4) of the vertex operator $V_{m,n}$ in the Gaussian model. Comparing Eq. (3.6) with Eq. (2.5), it is obvious that Q must be an integer, so N must be even, and

$$L_b \left(r = \left[2 \left(1 - \frac{\gamma}{\pi} \right) \right]^{-1/2} \right) \Leftrightarrow H_{XXZ}^{cont}(\gamma). \quad (3.8)$$

So, in Eq. (3.8) is given the first half of equivalence between Eq. (3.1) and the SGM, that unperturbed CFT's are equivalent. Now one must show the second part, that operator σ_n^x is the lattice counterpart of $V_{1,0}^{(+)}(x)$ ($x = na$) in the Gaussian model. In [23] it was shown (in the leading order in the lattice constant a) that

$$\sigma_n^{\pm} \propto a^{d_{1,0}} V_{\pm 1,0}(x) = a^{\beta^2/4\pi} V_{\pm 1,0}(x), \quad (3.9)$$

where $x = na$. The constant of proportionality in Eq. (3.9) is in fact known [24,25] but we will not need it here. So, from Eq. (3.9) we see that

$$\sigma_n^x \propto V_{1,0}^{(+)}(x), \quad x = na \quad (3.10)$$

in the leading order. That finally completes the argument [14] that Hamiltonian (3.1) is a spin chain regularization of the SGM where connection between coupling constants is

$$\beta = \frac{\sqrt{\pi}}{r} = \sqrt{2(\pi - \gamma)}. \quad (3.11)$$

Let us make a comment on internal symmetries of continuum and lattice models. As we emphasized in the last section SGM possesses $Z_2 \times \tilde{Z}_2 \times \tilde{U}(1)$ symmetry and is integrable. But spin chain (3.1) is only symmetric on Z_2 generated by “charge conjugation operator” C :

TABLE III. The same as Table II but now for $\Delta = -0.6$ ($\beta^2 = 4.43$).

h	\tilde{G}_{B1}	\tilde{G}_S	\tilde{G}_A	\tilde{G}_{B2}
0.8	4.48354(1)	5.9727(1)	7.477(1)	8.305(4)
0.5	4.51002(3)	6.199(1)	7.386(6)	8.41(1)
0.3	4.537(1)	6.38(1)	7.28(3)	8.49(5)
0.2	4.548(5)	6.47(3)	7.16(7)	8.56(13)

TABLE IV. The same as Table II but now for $\Delta = -0.1$ ($\beta^2 = 3.34$).

h	\tilde{G}_{B1}	\tilde{G}_S	\tilde{G}_A	\tilde{G}_{B2}
0.8	3.795834(2)	7.21140(8)	7.7036(2)	7.261(5)
0.5	3.75549(3)	7.483(1)	7.715(2)	7.21(1)
0.3	3.7372(3)	7.63(1)	7.73(1)	7.16(1)
0.2	3.728(1)	7.65(3)	7.71(4)	7.11(2)

$$C = \prod_{n=1}^N \sigma_n^x$$

and in fact is believed to be nonintegrable. That spin chain representation of a QFT has less symmetries is not something new [17].

Now, what are the relations between dimensionful parameters (L, λ, μ) in the (continuum) SGM and parameters (N, h) in (lattice) (3.1)? From Eqs. (3.7a) and (3.8) follows

$$H_{SGM}(L) = \frac{1}{\zeta} \lim_{\substack{N \rightarrow \infty \\ a = L/N}} \frac{H}{a}.$$

So, if we denote by \tilde{m}_i mass gaps in the spin chain, we have

$$m_i(L) = \frac{1}{\zeta} \lim_{\substack{N \rightarrow \infty \\ a = L/N}} \frac{\tilde{m}_i}{a}. \quad (3.12)$$

Also, from Eq. (3.10) we have

$$h \propto \lim_{a \rightarrow 0} \lambda a^{d_\lambda} = \lim_{a \rightarrow 0} \lambda a^{2 - \beta^2/4\pi}, \quad (3.13)$$

where the factor of proportionality is finite. Of course, we have $L = Na$ and λ fixed. We can see from Eq. (3.13) that $h \rightarrow 0$ because $d_\lambda > 0$. We can now express scaling parameter μ using lattice constants:

$$\mu = \lambda L^{d_\lambda} \propto \lim_{\substack{N \rightarrow \infty \\ L, \lambda \text{ finite}}} h N^{d_\lambda}. \quad (3.14)$$

Constant of proportionality is not important for us because we are interested here only in $L \rightarrow \infty$ ($\mu \rightarrow \infty$) and $L \rightarrow 0$ ($\mu \rightarrow 0$) limits. If we define now

$$\tilde{\mu} \equiv h N^{d_\lambda} = h N^{2 - \beta^2/4\pi} = h N^{3/2 + \gamma/2\pi} \quad (3.15)$$

TABLE V. Estimates for the mass gap ratios $\tilde{r}_a(\Delta, h)$ as a function of h at $\Delta = -1$ ($\beta^2 = 2\pi$). We also added predictions obtained from Eq. (1.1) (DHN) and Eq. (1.2) (Fujita *et al.*).

\tilde{r}_a	h				DHN	Fujita
	0.8	0.5	0.3	0.2		
S	1	1	1	1	1	0.877
A	1.4703(7)	1.419(4)	1.36(1)	1.32(2)	1	0.877
B2	1.762(2)	1.766(7)	1.74(2)	1.62(5)	1.732	

TABLE VI. The same as Table V but now for $\Delta = -0.9$ ($\beta^2 = 5.38$).

\tilde{r}_a	h				DHN	Fujita
	0.8	0.5	0.3	0.2		
S	1.07577(3)	1.0862(2)	1.095(3)	1.101(7)	1.205	1.018
A	1.5142(5)	1.467(3)	1.42(1)	1.37(3)	1.205	1.018
B2	1.792(1)	1.807(7)	1.84(3)	1.73(5)	1.820	

from Eqs. (2.9), (3.12), (3.14) and (3.15) we can see that

$$\tilde{m}_i = h^{(2-\beta^2/4\pi)^{-1}} \tilde{G}_i(\gamma, \tilde{\mu}), \quad (3.16)$$

where γ is connected to β by Eq. (3.11). Strictly speaking, scaling law (3.15) should be exactly valid only in the continuum limit $N \rightarrow \infty$, $a \rightarrow 0$ and $h \rightarrow 0$ where $L = Na$ and $\lambda \propto ha\beta^{2/4\pi-2}$ are kept fixed. For finite N , Eq. (3.15) is only approximate and we expect that scaling is worse for smaller N .

To keep our promise, we shall now comment on subtraction of “nonuniversal momentum” π mentioned in the part of the text following Eq. (3.7b), which does not sound very natural (maybe “too statistical”). A more natural explanation is based on the fact that SGM is equivalent to Eq. (3.1) when the number of lattice sites N is *even*. Let us suppose that the lattice is staggered, i.e., that (in continuum limit terms) real space translations are given by translations by *even* number of sites, and translation by one site is some internal state transformation [26]. A consequence is that T^2 is the “real” lattice translation operator, so $2P$ is the “real” momentum which is also defined mod 2π . But, now we must multiply Eq. (3.5b) by 2, so how can we obtain the same conformal dimensions Δ and $\bar{\Delta}$. An explanation is that the continuum spatial extension of the system is now $L = aN/2$, so we must put $N/2$ in place of N in Eq. (3.5a). In Eq. (3.5b) it just compensates factor 2, and in Eq. (3.5a) we already needed scaling factor ζ which should now be halved.

IV. MASS SPECTRUM

Now we are ready to calculate particle mass ratios in the SGM $L \rightarrow \infty$ limit using connection with spin chain (3.1). First we must numerically calculate mass gaps of spin chain for finite N and h . Then we must make a continuum limit, i.e., take $N \rightarrow \infty$ keeping $L = Na$ and $\tilde{\mu}$ fixed [obviously $a \rightarrow 0$ and from Eq. (3.15) $h \rightarrow 0$]. Finally we should make a

TABLE VII. The same as Table V but now for $\Delta = -0.6$ ($\beta^2 = 4.43$).

\tilde{r}_a	h				DHN	Fujita
	0.8	0.5	0.3	0.2		
S	1.33214(3)	1.3745(2)	1.406(3)	1.423(7)	1.517	1.229
A	1.6677(2)	1.638(1)	1.605(8)	1.57(2)	1.517	1.229
B2	1.8523(9)	1.865(3)	1.87(1)	1.88(3)	1.888	

TABLE VIII. The same as Table V but now for $\Delta = -0.4$ ($\beta^2 = 3.96$).

\tilde{r}_a	h				DHN	Fujita
	0.8	0.5	0.3	0.2		
S	1.53365(3)	1.5970(2)	1.639(3)	1.654(8)	1.724	1.367
A	1.7927(1)	1.779(1)	1.762(6)	1.74(1)	1.724	1.367
B2	1.880(1)	1.886(3)	1.885(5)	1.90(2)	1.914	

$L \rightarrow \infty$, i.e., $\tilde{\mu} \rightarrow \infty$ [see Eq. (3.14)] limit. In practice, it is preferable to do the following [17–19]: first take $N \rightarrow \infty$ with h fixed and afterwards extrapolate to $h \rightarrow 0$. A difference is that in the latter case one does $\tilde{\mu} \rightarrow \infty$ before $h \rightarrow 0$. These limits are performed using the BST extrapolation method [15,16].

We numerically diagonalized Hamiltonian (3.1) for up to 16 sites using the Lanczos algorithm. But before doing numerics, one should maximally exploit symmetries. The Hamiltonian (3.1) commutes with translation operator T [given by Eq. (3.3)] and with charge conjugation operator C . So, we can break Hamiltonian (3.1) in blocks, each marked with eigenvalues of the operators $P = i\ln T$ and C which can be $P_k = (2\pi i/N)k \bmod 2\pi$ [see Eq. (3.4)] and $C = \pm 1$ (because $C^2 = 1$). We are interested in mass ratios, so we only need zero-momentum sector. But, because “true” space translations are generated by T^2 (or because we must subtract “nonuniversal macroscopic momentum” π , if you like it more) zero-momentum sector is a union of $P=0$ and $P=\pi$ sectors. So we must diagonalize four blocks which we will denote by 0^+ , 0^- , π^+ and π^- .

We considered a number of values of coupling $-1 \leq \Delta < 1$ [or $\sqrt{2}\pi \geq \beta > 0$, see Eq. (3.11)]. Starting from $\Delta = -1$ the spectrum contains five clearly isolated states: vacuum and second breather in 0^+ , first breather in 0^- , soliton in π^- and antisoliton in π^+ . All other levels form “continuum,” i.e., they “densely” fill the region between $\approx 2 \times (\text{mass of first breather})$ and some E_{\max} . Soliton and antisoliton energies are not degenerate which is a consequence of breaking \tilde{Z}_2 symmetry on the spin chain. Exactly at $\Delta = -1$ we have [27] $\tilde{m}_{B1} = \tilde{m}_S < \tilde{m}_A < \tilde{m}_{B2}$. As we increase Δ , \tilde{m}_S , \tilde{m}_A and \tilde{m}_{B2} monotonically increase (relative to \tilde{m}_{B1}) where \tilde{m}_S and \tilde{m}_A increase faster than \tilde{m}_{B2} and at $\Delta \approx -0.1$ disappear into the “continuum” (i.e., $\tilde{m}_{S,A} > 2\tilde{m}_{B1}$), while \tilde{m}_{B2} asymptotically approach $2\tilde{m}_{B1}$. This

TABLE IX. The same as Table V but now for $\Delta = -0.1$ ($\beta^2 = 3.34$).

\tilde{r}_a	h				DHN	Fujita
	0.8	0.5	0.3	0.2		
S	1.89982(2)	1.9926(3)	2.042(4)	2.052(9)	2.096	1.612
A	2.02949(8)	2.0543(7)	2.068(3)	2.07(1)	2.096	1.612
B2	1.913(1)	1.920(3)	1.916(4)	1.907(7)	1.942	

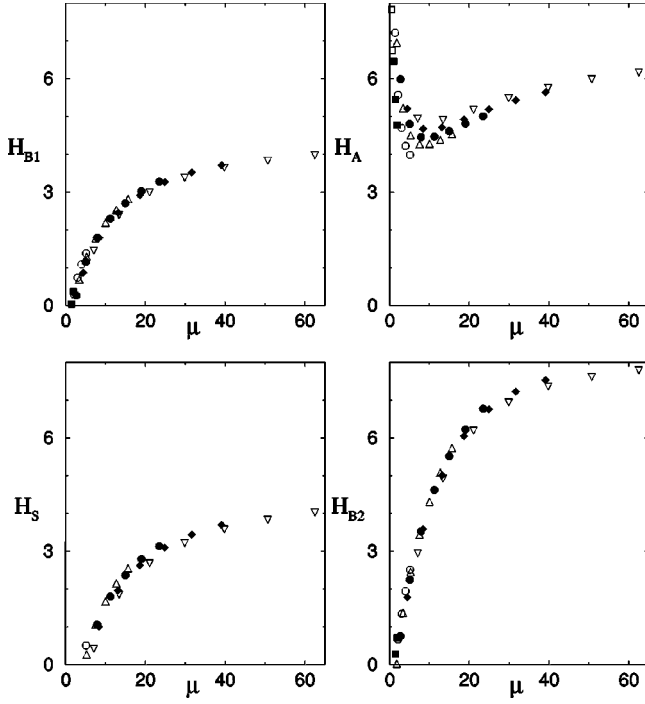


FIG. 4. Reduced scaling functions $\tilde{H}_a(\beta, \mu)$ at $\Delta = -0.9$ (or $\beta^2 = 5.38$). A legend is the same as in Fig. 1.

was a crude picture visible already from row data before extrapolation $N \rightarrow \infty$ and $h \rightarrow 0$, and it is expected from the DHN formula (1.1). Observe that the exact degeneracy of soliton and first breather masses at $\Delta = -1$ is present in Eq. (1.1).

In Figs. 1–3 we present numerical results for the scaled gaps (scaling functions of mass gaps) $\tilde{G}_a, a \in \{S, A, B1, B2\}$ at $\Delta = -0.9, -0.6, -0.1$. This is of course a check of the scaling relation (3.16). BST extrapolations $N \rightarrow \infty$ (with fixed h) of scaled gaps for $h = 0.8, 0.5, 0.3, 0.2$ are given in Tables II–IV. As expected convergence is better for higher Δ .

To make an extrapolation $h \rightarrow 0$ one should obtain results for smaller h , at least $h \approx 0.1$. From Figs. 1–3 one can see that for that one should diagonalize the Hamiltonian with $N \geq 26$, which is too demanding even for the most powerful machines today.

Finally, (partially) extrapolated mass ratios

$$\tilde{r}_a(\Delta, h) = \lim_{\substack{N \rightarrow \infty \\ h \text{ fixed}}} \frac{\tilde{m}_a}{\tilde{m}_{B1}} = \lim_{\substack{N \rightarrow \infty \\ h \text{ fixed}}} \frac{\tilde{G}_a}{\tilde{G}_{B1}}, \quad a \in \{S, A, B2\}$$

are given in Tables V–IX together with the predictions from DHN formula (1.1) and Fujita *et al.* formula (1.2). One can see that our results confirm DHN and reject Fujita *et al.*

V. UV (CONFORMAL) LIMIT OF PARTICLE STATES

Let us now turn our attention to the opposite UV limit of our results for the spin chain (3.1). We saw in Sec. II that it is obtained when $\mu(\tilde{\mu}) \rightarrow 0$. Using Eqs. (3.12) and (3.15) in the continuum result (2.11) we obtain that the scaling rela-

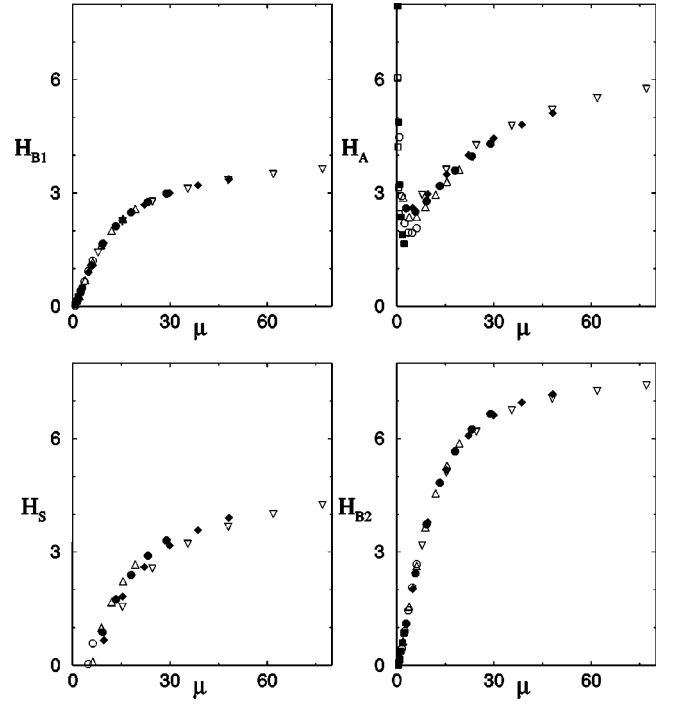


FIG. 5. The same as Fig. 4 but now for $\Delta = -0.6$ (or $\beta^2 = 4.43$).

tion for mass gaps of spin chain should have the form

$$\tilde{m}_a(\gamma, \tilde{\mu}, h) = \zeta h^{2\pi/(3\pi+\gamma)} [2\pi d_a \tilde{\mu}^{-2\pi/(3\pi+\gamma)} + \tilde{H}_a(\gamma, \tilde{\mu})],$$

where we must now include proper normalization factor ζ

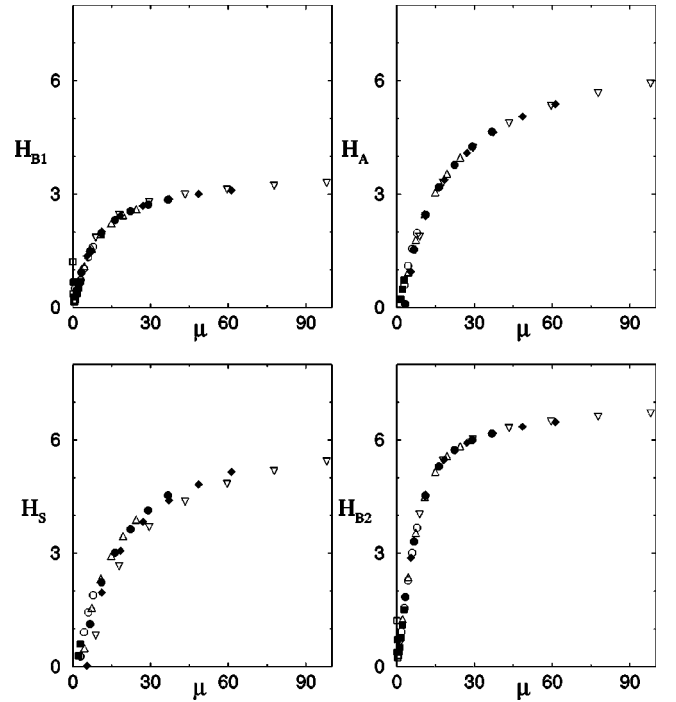


FIG. 6. The same as Fig. 4 but now for $\Delta = -0.1$ (or $\beta^2 = 3.34$).

TABLE X. Scaling dimensions of particle states in SGM as conjectured from our numerical results.

State	Operator	Scaling dimension
soliton	$V_{0,1}$	$\frac{\pi}{\beta^2} = \frac{1}{2} \left(1 - \frac{\gamma}{\pi}\right)^{-1}$
antisoliton	$V_{0,-1}$	$\frac{\pi}{\beta^2} = \frac{1}{2} \left(1 - \frac{\gamma}{\pi}\right)^{-1}$
1st breather	$V_{1,0}^{(-)}$	$\frac{\beta^2}{4\pi} = \frac{1}{2} \left(1 - \frac{\gamma}{\pi}\right)$
2nd breather	$V_{1,0}^{(+)}$	$\frac{\beta^2}{4\pi} = \frac{1}{2} \left(1 - \frac{\gamma}{\pi}\right)$

for the spin chain Hamiltonian. Because it does not depend on h we can take it from unperturbed XXZ spin chain (3.2), where it is well known

$$\zeta = \frac{\pi \sin \gamma}{\gamma}.$$

Before we plot reduced scaling functions $\tilde{H}_a(\gamma, \tilde{\mu})$ we must know scaling dimension d_a of the corresponding state. On the other hand, we can choose d_a and see if it gives the right behavior of $\tilde{H}_a(\gamma, \tilde{\mu})$ when $\tilde{\mu} \rightarrow 0$ [which is the same as for H_a mentioned below Eq. (2.10)].

In Table I we have presented scaling dimensions of zero-

momentum particle states of SGM as conjectured in [14]. But our numerical results clearly indicate that the first and second breather ($B2$ and $B2$) have exactly the same scaling dimensions. In Figs. 4–6 we show numeric results for reduced scaling functions, where we used values from Table X for scaling dimensions.

We can see in Figs. 4–6 that finite size effects are stronger for Δ closer to -1 (where they are in fact logarithmic because of the appearance of marginal operators), which is expected from [28].

VI. CONCLUSION

In this paper we use the XXZ spin chain in a transverse field as a lattice regularization of the sine-Gordon model proposed in [14]. This equivalence can be understood, e.g., from conformal perturbation theory. One of our goals was to calculate by numerical analysis masses in the sine-Gordon theory. This is now of interest because recent calculations based on numerical treatment of the Bethe ansatz [11] and infinite momentum frame technique [10] are in disagreement with previous approaches used in literature [4–7]. Our results are in agreement with the DHN formula contrary to previously mentioned papers. We stress that methods used in this paper are independent of previous approaches to SGM (which were criticized in [10–12]). We also analyze the conformal limit and find conformal dimensions of various states. We find that the conformal dimension of the second breather state disagrees with the conjecture by [14]. Our calculations for dimensions of other states agree with those in [14].

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